Table 1 Closed-loop poles for $R_r = 1.0$ ($r = 1,2,3$)

Case 1 $\alpha=\beta=0$	Case 2 $\alpha=\beta=0.0001$	Case 3 $\alpha=\beta=0.00$
$-0.6199 \pm i1.0057$	$-0.6200 \pm i1.0056$	$-0.6211 \pm i1.0050$
$-0.6921 \pm i1.8436$	$-0.6936 \pm i1.8430$	$-0.7076 \pm i1.8382$
$-0.7037 \pm i3.6792$	$-0.7112 \pm i3.6778$	$-0.7815 \pm i3.6676$
$\pm i6.3165$	$-0.0231 \pm i6.3168$	$-0.2248 \pm i6.3379$
$\pm i9.8696$	$-0.0562 \pm i9.8704$	$-0.5643 \pm i9.9485$
$\pm i14.2122$	$-0.1162 \pm i14.2146$	$-1.1630 \pm i14.4489$
$\pm i19.3444$	$-0.2148 \pm i19.3504$	$-1.8443 \pm i20.4288$
$\pm i25.2662$	$-0.3360 \pm i25.2453$	$-3.9635 \pm i22.6735$

250.4, and 252.3 corresponding to cases 1, 2, and 3, respectively. Hence, damping may enhance or degrade control system performance depending on the value of the gains. As the gains increase, larger amounts of energy are transferred back and forth between the controlled and uncontrolled modes, which are coupled due to the presence of damping. This effect is known as control spillover. However, we also conclude that the damping does not alter control system performance significantly.

Conclusions

Quite often, control system design is based on the undamped self-adjoint distributed structure. Inherent structural damping tends to destroy the self-adjointness of the structure and can degrade control system performance. However, depending on the control system design, the effect of small damping can be negligible. Indeed, when control is designed by the independent modal space control method (IMSC), the perturbation in the closed-loop poles due to small damping does not cause a dramatic shift in the closed-loop poles nor does it change the control system performance very much. The sensitivity study presented here for IMSC indicates that damping shifts the actual closed-loop poles to the left of the modeled closed-loop poles in the complex plane. Moreover, in the case of IMSC, the presence of small damping in the actual distributed structure does not affect the control system performance significantly.

Acknowledgments

Supported in part by the USAF/ASD and the Air Force Office of Scientific Research Grant F33615-86-C-3233, monitored by Drs. A. K. Amos and V. B. Venkayya, whose support is greatly appreciated.

References

[1]Meirovitch, L., *Computational Methods in Structural Dynamics,* Sijthoff & Noordhoff, the Netherlands, 1980.

[2]Meirovitch, L. and Baruh, H., "Control of Self-Adjoint Distributed-Parameter Systems," *Journal of Guidance and Control,* Vol. 5, Jan.-Feb. 1982, pp. 60-66.

[3]Clough, R. W. and Penzien, J., *Dynamics of Structures,* McGraw-Hill, New York, 1975.

Application of Output Feedback to Variable Structure Systems

Bonnie S. Heck* and Aldo A. Ferri†
Georgia Institute of Technology, Atlanta, Georgia

I. Introduction

VARIABLE structure systems (VSS), especially those with a sliding mode, have generated considerable interest recently in control-theory applications due to the benefits achieved: good robustness, disturbance rejection, model reduction, and possible linearization through the control. This control scheme has been applied in chemical and industrial processes, flight control, spacecraft attitude control, robotics, and motor control.[1-4] More information and a literature survey may be found in the papers by Utkin.[1,5]

The use of VSS has been previously limited to systems with full-state feedback. In practice, however, full measurement of the state vector is either not possible or not feasible. Instead, only certain outputs are measured. Recently, asymptotic observers have been used in VSS to reconstruct the state vector for estimated state feedback.[6,7] This adds dynamics to the compensator, which increases the complexity in the implementation. In fact, it may be physically unrealizable for a large-order system. Also, convergence to the sliding manifold is asymptotic as $t \to +\infty$, and so the invariance properties found with state feedback do not strictly hold. Even if the states are all measurable, implementation with state feedback is necessarily complex, requiring many feedback loops. The purpose of this Note is to introduce output feedback as an alternative to state feedback and to estimated state feedback for simplification of control-system implementation.

Output feedback used in linear systems has generated much interest in the last 20 years (see, for example, Refs. 8-10). The main benefits are that there are no dynamics in the compensator and there are fewer feedback loops. There are, however, inherent problems with output feedback in linear systems that are also present when applied to VSS. For example, it may not be possible to stabilize a system with a given set of outputs using static output feedback. Also, output feedback is generally difficult to design. However, in many cases, the simplification in implementation far outweighs the difficulties in design.

The outline of this Note is as follows. Section II contains the basic considerations of output feedback in VSS. Section III discusses two design procedures. An example illustrating the technique is given in Sec. IV, and the conclusions are given in Sec. V.

II. Output Feedback in VSS

The system under consideration is time-invariant and can be represented in the following form

$$\dot{x} = f(x) + B(x)u \tag{1}$$

$$y = Cx \tag{2}$$

where $x \in R^n$, $u \in R^m$, $y \in R^p$, f and B satisfy the Lipschitz conditions, and C is a matrix. [For simplicity of notation, $f(x)$ will be denoted as f, and $B(x)$ will be denoted as B.] The components of the control vector u are given as a function of the output y, i.e.,

$$u_i = \begin{cases} u_i^+(y) & \text{if } s_i(y) > 0 \\ u_i^-(y) & \text{if } s_i(y) < 0 \end{cases}, \qquad i = 1,\dots,m \tag{3}$$

where each s_i is a linear functional, and each u_i^+ and u_i^- satisfies a Lipschitz condition. A sliding mode may occur on any of the surfaces defined by $s_i(y) = 0$ or along any intersection of these surfaces. This Note will address the case of sliding along the surface $s(y) = [s_1,\dots,s_m]^T(y) = 0$, although the other cases are easily extracted from this analysis.

Using the equivalent control method,[1] the equation of motion for the system [Eqs. (1) and (2)] in the sliding mode is given by:

$$\dot{x} = f - B(GCB)^{-1}GCf \tag{4}$$

where $s = Gy$. Note that Eq. (4) is valid only when $s = 0$. The switching function $s = Gy$ is chosen to stabilize the sliding-mode equation under the constraint that GCB be invertible.

Received Jan. 8, 1988; revision received Aug. 1, 1988.

*Assistant Professor, School of Electrical Engineering.
†Assistant Professor, School of Mechanical Engineering. Member AIAA.

(This condition is needed for existence and uniqueness of the solution.[1]) Thus, for a linear system, CB must have full rank. An obvious subclass of this type of system is one where each actuator is collocated with a sensor.

Aside from the condition that GCB be invertible, the choice of the sliding surface is not as arbitrary as it is with full-state feedback. The sliding surface in the measurement space (R^p) is specified by $s = Gy = 0$, i.e., it is the null space of G. Similarly, the sliding surface viewed in the state space is the null space of GC. The effect of the output feedback in the state space is that the orientation of the sliding surface is restricted and cannot be chosen freely. In particular, the null space of GC is spanned by $n - m$ n-dimensional vectors, $n - p$ of which are determined by the null space of C. Therefore, only $p - m$ spanning vectors may be chosen freely. Thus, if $p = m$, the sliding surface and the dynamics of the sliding mode are predetermined regardless of G. In that case, the sliding surface is equal to the null space of C. The corresponding sliding-mode equation (4) is

$$\dot{x} = f - B(CB)^{-1}Cf \tag{5}$$

If $p = n$ and C has full rank, then the sliding surface and the dynamics of the sliding mode can be chosen with the same amount of freedom as in the full-state feedback case. It follows that since C is invertible, the dimension of the null space of GC is equal to the dimension of the null space of G. Defining $G' = GC$, the sliding-mode equation (4) becomes

$$\dot{x} = f - B(G'B)^{-1}G'f \tag{6}$$

where, as in the state feedback case, G' may be chosen arbitrarily, subject only to the constraint that $G'B$ be invertible. In most applications $m < p < n$, and so some freedom remains in choosing the sliding surface, while it remains more restrictive than full-state feedback.

The invariance properties of sliding-mode control, which yield good robustness and disturbance rejection, are directly achievable with output feedback. It is straightforward to show that the analysis in Ref. 11 for the invariance properties of state feedback VSS also applies to output feedback by substituting $s(x) = GCx$. Hence, the good robustness and good disturbance-rejection benefits of VSS are present when the control is implemented with output feedback.

One of the most common applications of VSS theory is in linear systems in regular form. The use of output feedback in such an application is investigated by considering the following system:

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 \tag{7a}$$

$$\dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_2u \tag{7b}$$

$$y = C_1x_1 + C_2x_2 \tag{7c}$$

where $x_2 \in R^m$, $\boldsymbol{u}$ is as given in Eq. (3), and y is the measurement. Assuming sliding to exist on the $s = Gy = 0$ switching surface, the sliding-mode equation may be reduced in order to the following $(n - m)$-dimensional form.

$$\dot{x}_1 = [A_{11} - A_{12}(GC_2)^{-1}GC_1]x_1 \tag{8}$$

(The assumption that GC_2 is invertible follows from the requirement that GC_2B_2 be invertible.) The switching surface $s = Gy = 0$ is determined by choosing G to stabilize the reduced-order system [Eq. (8)].

III. Design Procedures

The two aspects to design of VSS are selecting the sliding surface to ensure stability of the sliding mode and selecting the control components to ensure existence (or reachability) of the sliding surface. Stability of the sliding mode may not be achievable with output feedback. This is seen most readily when the number of outputs equals the number of inputs. In this case, no control-design parameters exist [see Eq. (5)], and so stability depends only on the original system parameters. If a system cannot be stabilized with a given set of outputs, the outputs must be changed and/or additional outputs used. Existence of the sliding surface can be analyzed using state space methods shown in Ref. 1 where $s(x) = GCx$. The existence condition (i.e., local stability of the sliding mode) can be satisfied by setting $\boldsymbol{u}^+$ and $\boldsymbol{u}^-$ equal to constant values as done with a bang-bang control. The reaching condition (i.e., global stability of the sliding surface) is, in general, difficult to guarantee with output feedback. Descriptions for two design procedures follow, one for stabilizing the sliding mode and one for guaranteeing that the reaching condition is met for a particular class of systems.

The selection of G to stabilize the sliding-mode equation (8) may be done using standard pole assignment algorithms for output feedback (e.g., Ref. 10), modified by an additional constraint equation. Defining $A = A_{11}$, $B = A_{12}$, $C = C_1$, and $K = (GC_2)^{-1}G$, Eq. (8) is rewritten as

$$\dot{x}_1 = (A - BKC)x_1 \tag{9}$$

The selection of K to stabilize Eq. (9) represents the standard linear output feedback problem. The additional constraint is that there must exist a G such that $K = (GC_2)^{-1}G$. A necessary and sufficient condition for this is

$$\text{rank}\,[C_2K - I] \leq p - m \tag{10}$$

Note that this condition must be met solely if pole placement algorithms are to be used. If only stabilization is desired, the conditions are less restrictive.

The following design procedure for output feedback guarantees that the reaching condition is met for a general class of linear systems. Consider the general time-invariant linear system:

$$\dot{x} = Ax + Bu \tag{11a}$$

$$y = Cx \tag{11b}$$

The control $\boldsymbol{u}$ is chosen as

$$\boldsymbol{u} = -(GCB)^{-1}GCANy - (GCB)^{-1}\frac{s}{\|s\|^2} \tag{12}$$

where $s = Gy$. This control law is a modification of a state feedback control law proposed by DeCarlo et al.[12] The reaching condition is met if $s^T\dot{s} < 0$ for all x.[1] The following expression for $s^T\dot{s}$ can be obtained:

$$s^T\dot{s} = x^TC^TG^TGCA(I - NC)x - \frac{s^Ts}{\|s\|^2} \tag{13}$$

If N can be chosen such that the following condition is satisfied

$$C^TG^TGCA(I - NC) \leq 0 \tag{14}$$

(i.e., the matrix is negative semidefinite), then $s^T\dot{s} \leq -1$ so that the reaching condition is met. The general class of systems for which this control law guarantees that the reaching condition is met is any linear system for which an N exists satisfying Eq. (14). For ease of design, if $C^TG^TGCA \leq 0$, then $N = 0$ is sufficient. However, N influences the time to reach the sliding surface and may be chosen accordingly.

For practical reasons, it is desirable to limit the control authority in Eq. (12) since as $s \to 0$, $\boldsymbol{u} \to \pm\infty$. Hence, the following control law is proposed:

$$\boldsymbol{u} = -(GCB)^{-1}GCANy - (GCB)^{-1}g(s) \tag{15}$$

where

$$g(s) = \begin{bmatrix} \text{sat}(s_1/\|s\|^2) \\ \text{sat}(s_2/\|s\|^2) \\ \vdots \\ \text{sat}(s_m/\|s\|^2) \end{bmatrix}, \qquad \text{sat}(z) = \begin{cases} a_i, & z > a_i \\ z, & |z| \le a_i \\ -a_i, & z < -a_i \end{cases}$$

and each $a_i > 0$. If the condition in Eq. (14) is satisfied, then

$$s^T\dot{s} \le -s^T g(s) \tag{16}$$

When $\|s\|$ is large, $g(s)$ does not saturate and $s^T\dot{s} \le -1$. However, when $\|s\|$ is small

$$s^T\dot{s} \le -\sum_{i=1}^{m} a_i |s_i| \le 0 \tag{17}$$

Again, the reaching condition is met, insuring that the sliding surface is globally stable.

IV. Example

An example of output feedback in VSS follows for a linearized model of the longitudinal dynamics of an aircraft. The states are the angle of attack α, pitch rate q, and elevator angle δ_e, and the control input u is the command to the elevator. The system is given by:

$$\begin{bmatrix} \dot{\alpha} \\ q \\ \delta_e \end{bmatrix} = \begin{bmatrix} -0.277 & 1 & -0.0002 \\ -17.1 & -0.178 & -12.2 \\ 0 & 0 & -6.67 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \delta_e \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 6.67 \end{bmatrix} u \tag{18}$$

The measurements are chosen to be pitch rate and elevator angle; the latter is chosen for collocation of sensor and actuator. Hence, the measurement vector y is defined as

$$y = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \delta_e \end{bmatrix} \tag{19}$$

This system is easily put into regular form [Eqs. (7)] by defining $x_1 = [\alpha \;\; q]^T$ and $x_2 = \delta_e$, and identifying the parameter matrices from Eqs. (18) and (19) as

$$A_{11} = \begin{bmatrix} -0.277 & 1 \\ -17.1 & -0.178 \end{bmatrix}, \qquad A_{12} = \begin{bmatrix} -0.0002 \\ -12.2 \end{bmatrix}$$

$$A_{21} = [0 \;\; 0], \qquad A_{22} = [-6.67], \qquad B_2 = [6.67]$$

$$C_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{20}$$

The switching surface is selected as follows, using the pole assignment technique discussed previously. The system is put into the form of Eq. (9) where $K = [k_1, k_2]$. The constraint in Eq. (10) requires that

$$\text{rank} \begin{bmatrix} -1 & 0 \\ k_1 & k_2 - 1 \end{bmatrix} \le 1 \tag{21}$$

Hence, $k_2 = 1$. Because of the structure of the matrices in Eq. (9), only k_1 affects the eigenvalues, and so the constraint on k_2 puts no additional restriction on the placement of the eigenvalues. It is found that a choice of $K = [-0.4635, 1]$ places the eigenvalues at

$$\lambda_{1,2} = -3.06 \pm 3.06j \tag{22}$$

Solving for G yields $G = [-0.4635, 1]$. Note that arbitrary pole placement is not possible, as only a single design parameter k_1 exists.

The control is chosen in the form of Eq. (15) (where $B = [0^T B_2^T]^T$ and $C = [C_1 \; C_2]$) to insure global stability of the sliding surface. For this system, $C^T G^T G C A$ is negative semidefinite, and so N is chosen to be 0. Hence, the control is of the form

$$u = -(GCB)^{-1} g(s) \tag{23}$$

where

$$g(s) = \begin{cases} 3.34, & s/|s|^2 > 3.34 \\ s/|s|^2, & 1/|s| \le 3.34 \\ -3.34, & s/|s|^2 < -3.34 \end{cases}$$

The saturation is chosen so that the command to the elevator angle is limited to $|u| \le 0.5$.

The time response of the system subject to an initial condition of $[0 \;\; 2 \;\; 0]^T$ is shown for α and q in Figs. 1 and 2, respectively. Sliding occurs at approximately $t = 0.28$ s, i.e.,

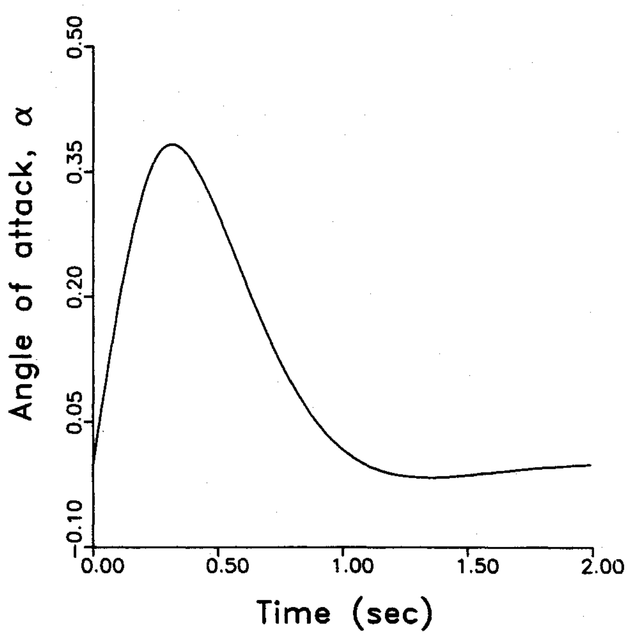

Fig. 1 Angle of attack vs time.

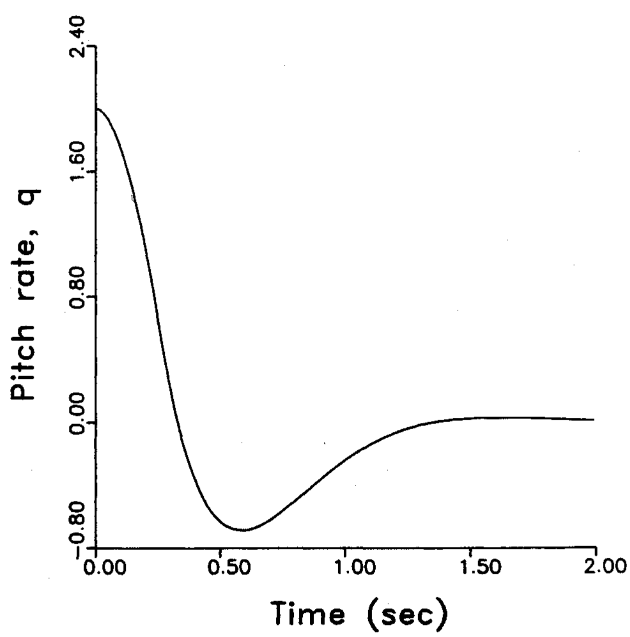

Fig. 2 Pitch rate vs time.

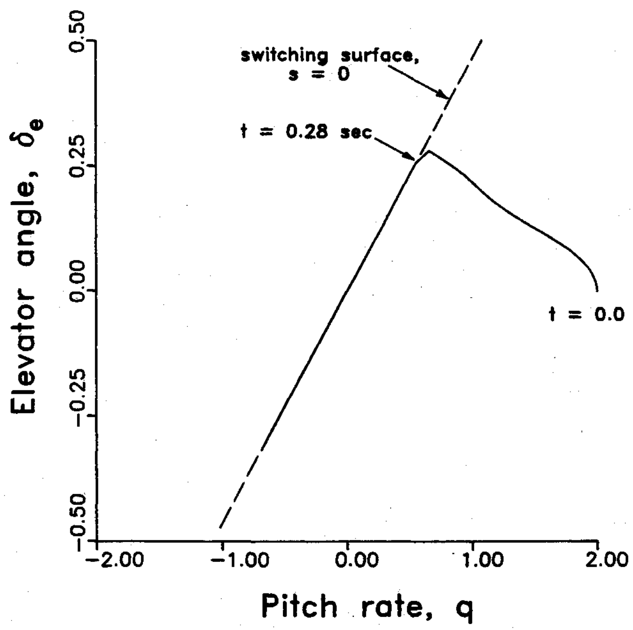

Fig. 3 Elevator angle vs pitch rate, trajectory in measurement space.

$s \simeq 0$ at this time. It should be noted that once sliding has occurred, the system behaves with second-order dynamics having eigenvalues specified by Eq. (22). In the sliding mode, the third state δ_e is simply given by a linear combination of α and q. Since $s = GC_1x_1 + GC_2x_2 = 0$,

$$\delta_e = -(GC_2)^{-1}GC_1\begin{bmatrix}\alpha\\ q\end{bmatrix} = 0.4635q \qquad (24)$$

In Fig. 3, the system trajectory is plotted in the measurement space y_2 vs y_1. The trajectory is seen to converge to the line $Gy = 0$, shown dashed in the figure.

V. Conclusions

In this Note, output feedback has been successfully applied to variable structure systems. The method allows one to use sliding-mode control theory on systems for which full-state feedback or estimated state feedback is not possible or not desirable. Output feedback allows for much simpler implementation, while at the same time providing good robustness to disturbance and plant uncertainty. Like output feedback in linear systems, it is seen that output feedback in variable structure systems is somewhat more restrictive and is harder to design than estimated state feedback with an asymptotic observer. It is also more sensitive to noise due to the direct link to the measurements. The ease of implementation, though, makes output feedback a viable alternative to either state feedback or estimated state feedback for many practical applications of variable structure systems.

References

[1]Utkin, V. I., "Variable Structure Systems with Sliding Modes," *IEEE Transactions on Automatic Control*, Vol. AC-22, April 1977, pp. 212–222.

[2]Young, K. D., "Controller Design for a Manipulator Using Theory of Variable Structure Systems," *IEEE Transactions on Systems, Man, and Cybernetics*, Vol. SMC-8, Feb. 1978, pp. 210–218.

[3]Calise, A. J. and Kramer, F. S., "A Variable Structure Approach to Robust Control of VTOL Aircraft," *Journal of Guidance, Control, and Dynamics*, Vol. 7, Sept.–Oct. 1984, pp. 620–626.

[4]Dwyer, T. A. W., III, and Sira-Ramirez, H., "Variable-Structure Control of Spacecraft Attitude Maneuvers," *Journal of Guidance, Control, and Dynamics*, Vol. 11, May–June 1988, pp. 262–270.

[5]Utkin, V. I., "Variable Structure Systems: Present and Future," *Automation and Remote Control*, Vol. 44, Sept. 1983, pp. 1105–1120.

[6]Bondarev, A. G., Bodarev, S. A., Kostyleva, N. E., and Utkin, V. I., "Sliding Modes in Systems with Asymptotic State Observers," *Automation and Remote Control*, Vol. 46, June 1985, pp. 679–684.

[7]Walcott, B. L. and Żak, S. H., "State Observation of Nonlinear Uncertain Dynamical Systems," *IEEE Transactions on Automatic Control*, Vol. AC-32, Feb. 1987, pp. 166–170.

[8]Levine, W. S., Johnson, T. L., and Athans, M., "Optimal Limited State Variable Feedback Controllers for Linear Systems," *IEEE Transactions on Automatic Control*, Vol. AC-16, Dec. 1971, pp. 785–793.

[9]Moerder, D. D. and Calise, A. J., "Convergence of a Numerical Algorithm for Calculating Optimal Output Feedback Gains," *IEEE Transactions on Automatic Control*, Vol. AC-30, April 1985, pp. 900–903.

[10]Davison, E. J., "On Pole Assignment in Linear Systems with Incomplete State Feedback," *IEEE Transactions on Automatic Control*, Vol. AC-15, June 1970, pp. 348–351.

[11]Drazenovic, B., "The Invariance Conditions in Variable Structure Systems," *Automatica*, Vol. 5, March 1969, pp. 287–295.

[12]DeCarlo, R. A., Żak, S. H., and Matthews, G. P., "Variable Structure Control of Nonlinear Multivariable Systems: A Tutorial," *Proceedings of the IEEE*, Vol. 76, March 1988, pp. 212–232.

Dynamics of a Rotationally Accelerated Beam

T. P. Mitchell* and J. C. Bruch Jr.*
University of California, Santa Barbara, Santa Barbara, California

Introduction

FLEXIBLE elements in non-steady-state rotation frequently constitute an important part of a larger mechanism. For example, a key area of research in robotics is the design of lightweight manipulators having quick response with modest energy input and whose dynamic response is not significantly affected by structural flexibility.

The work presented herein addresses the problem of finding the total displacement (that due to rigid-body motion plus flexibility effects) of a cantilevered beam subject to an arbitrary driving torque. The problem is formulated in a nondimensional manner and, thus, the solution is general. The results for a harmonic-applied torque are given explicitly in order to provide the basis for a Fourier synthesis of a more general torque expression.

Algorithms for analyzing the behavior of a cantilever beam built into a rigid base that performs specified motions in three dimensions have been developed by Kane et al.[1] Their paper also contains a bibliography of the extensive literature in this research area. In the context of robot dynamics the control of the beam dynamics has been investigated by, for example, Cannon and Schmitz,[2] Chassiakos and Bekey,[3] Hastings and Book,[4] and Rakhsha and Goldenberg.[5] In addition, Sakawa et al.[6] investigate the problem of a flexible arm rotated by a motor about an axis through the arm's fixed end. Several satisfactory experimental results are given. In the context of flexible space structures the control problem is addressed in a forthcoming paper by Wie and Bryson.[7] Their work was made known to the authors after the present paper had been completed and submitted for publication.

Received June 7, 1988; revision received Aug. 22, 1988.

*Professor, Department of Mechanical and Environmental Engineering.